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Characterizations of Continuous and Lipschitz Continuous Metric Selections in Normed Linear Spaces

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Characterizations are given of when the metric projection P_M onto a proximal subspace M has a continuous, pointwise Lipschitz continuous, or Lipschitz continuous selection. Moreover, it is shown that if P_M has a continuous selection, then it has one which is also homogeneous and additive modulo M. An analogous result holds if P_M has a pointwise Lipschitz or Lipschitz continuous selection provided that M is complemented. If $\dim M < \infty$ and P_M is Lipschitz (resp. pointwise Lipschitz) continuous, then P_M has a Lipschitz (resp. pointwise Lipschitz) continuous selection. A conjecture of R. Holmes and B. Kripke (*Michigan Math. J.* **15** (1968), 225–248) is resolved. © 1989 Academic Press, Inc.

1. INTRODUCTION

A (linear) subspace M of a normed linear space X is called *proximinal* (resp. Chebyshev) if, for each $x \in X$, the set of "best approximations" to x from M,

$$P_M(x) := \{ y \in M \mid \|x - y\| = \inf_{m \in M} \|x - m\| \}, \tag{1.1}$$

is nonempty (resp. a singleton). For example, any finite-dimensional subspace or any closed subspace in a reflexive space is proximinal, and a proximinal subspace in a strictly convex space is Chebyshev. Throughout the sequel, M is assumed to be proximinal. The set-valued mapping $P_M: X \to 2^M$ thus defined is called the *metric projection* onto M. A *selection* for P_M, or a *metric selection* for M, is any function $p: X \to M$ such that $p(x) \in P_M(x)$ for all $x \in X$. In this paper, we are mainly interested in

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selections which are also continuous, pointwise Lipschitz continuous, or Lipschitz continuous. (Conditions under which linear selections exist have been extensively studied in [2] and [9].) For much of what is known about these problems, the reader may consult the surveys [3] and [13].

Since we have to deal with set-valued mappings other than metric projections, we give the main definitions for a general set-valued map.

Let X and Y be (real) normed linear spaces and $F: X \to \mathscr{K}(Y)$, where $\mathscr{K}(Y)$ denotes the collection of all nonempty, closed, bounded, and convex subsets of Y. F is said to be *homogeneous* if

$$F(\alpha x) = \alpha F(x), \qquad x \in X,\ \alpha \in \mathbb{R}. \tag{1.2}$$

F is called *bounded* if there is a constant $c > 0$ such that

$$\sup\{\|y\| \mid y \in F(x)\} \leqslant c\,\|x\|, \qquad x \in X. \tag{1.3}$$

For example, if M is a proximinal subspace of X, then it is well known that $P_M: X \to \mathscr{K}(M)$ and P_M is homogeneous and bounded with constant $c = 2$. Moreover, P_M is "additive modulo M." That is,

$$P_M(x + m) = P_M(x) + m \tag{1.4}$$

for all $x \in X$, $m \in M$.

The *Hausdorff metric* h on $\mathscr{K}(Y)$ is defined by

$$h(A, B) := \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\},$$

where $d(x, A) := \inf\{\|x - a\| \mid a \in A\}$. The mapping $F: X \to \mathscr{K}(Y)$ is *pointwise Lipschitz continuous* if for each $x \in X$ there exists a constant $\lambda(x) > 0$ such that

$$h(F(x), F(y)) \leqslant \lambda(x)\,\|x - y\|, \qquad y \in X. \tag{1.5}$$

If in this definition the same constant λ works for all $x \in X$, F is called *Lipschitz continuous*. F is called *uniformly continuous* if for each $\varepsilon > 0$ there exists $\delta > 0$ so that $h(F(x), F(y)) < \varepsilon$ whenever x, y in X and $\|x - y\| < \delta$. F is called *lower semicontinuous* at $x \in X$ if $x_n \to x$ and $y \in F(x)$ implies $d(y, F(x_n)) \to 0$.

A selection p for P_M is said to be homogeneous or additive modulo M if it has this property regarded as a singleton-valued mapping, i.e.,

$$p(\alpha x) = \alpha p(x), \qquad x \in X,\ \alpha \in \mathbb{R} \tag{1.6}$$

or

$$p(x + m) = p(x) + m, \qquad x \in X,\ m \in M. \tag{1.7}$$

Finally, the *kernel* of the metric projection P_M is the set

$$\ker P_M := \{x \in X \mid 0 \in P_M(x)\}.$$

We can now outline some of the main results of this paper. In Section 2, it is noted that if P_M is Lipschitz (pointwise Lipschitz) continuous, then it admits a selection with the same property (Corollary 2.4). A conjecture of Holmes and Kripke [8] is also resolved.

In Section 3, characterizations are given for when P_M has a continuous (resp. pointwise Lipschitz continuous, Lipschitz continuous) selection which is homogeneous and additive modulo M (Theorem 3.3). Also, it is shown that P_M has a continuous selection if and only if P_M has a continuous selection which is homogeneous and additive modulo M (Theorem 3.4). If M is complemented, then P_M has a Lipschitz (resp. pointwise Lipschitz) continuous selection if and only if it has a selection of the same type which is homogeneous and additive modulo M (Theorem 3.5). A characterization is given of the proximinal subspaces of finite codimension which have continuous metric selections (Theorem 3.7). In the particular case when M is Chebyshev, a result of Cheney and Wulbert [1] is recovered (Corollary 3.10).

2. Lipschitz Continuous Metric Projections

Our first observation is that Lipschitz continuity and uniform continuity are the same for a certain class of set-valued mappings which include metric projections.

2.1. Proposition. *Let $F: X \to \mathscr{K}(Y)$ be bounded and homogeneous. Then the following statements are equivalent*:

(1) *F is Lipschitz continuous*;

(2) *F is uniformly continuous.*

Proof. (1) ⇒ (2) is obvious.

(2) ⇒ (1). Assume F is uniformly continuous. Then there exists $\delta > 0$ such that

$$h(F(x), F(y)) \leqslant 1 = \delta^{-1}\delta$$

whenever $\|x - y\| \leqslant \delta$. Setting $\lambda = \delta^{-1}$, we see that

$$h(F(x), F(y)) \leqslant \lambda\delta$$

whenever $\|x-y\| \leqslant \delta$. For any x, y in X with $x \neq y$, set

$$\tilde{x} = \frac{\delta x}{\|x-y\|}, \qquad \tilde{y} = \frac{\delta y}{\|x-y\|}.$$

Then $\|\tilde{x}-\tilde{y}\| = \delta$ implies

$$h(F(\tilde{x}), F(\tilde{y})) \leqslant \lambda\delta = \lambda\, \|\tilde{x}-\tilde{y}\|. \tag{2.1.1}$$

By the homogeneity of F and the positive homogeneity of h, we see that

$$\begin{aligned} h(F(\tilde{x}), F(\tilde{y})) &= h\left(\frac{\delta}{\|x-y\|} F(x), \frac{\delta}{\|x-y\|} F(y)\right) \\ &= \frac{\delta}{\|x-y\|} h(F(x), F(y)) \\ &= \frac{\|\tilde{x}-\tilde{y}\|}{\|x-y\|} h(F(x), F(y)). \end{aligned} \tag{2.1.2}$$

From (2.1.1) and (2.1.2) there follows

$$h(F(x), F(y)) \leqslant \lambda\, \|x-y\|.$$

Thus F is Lipschitz continuous. ∎

2.2. COROLLARY. *If M is a proximinal subspace of X, then P_M is Lipschitz continuous if and only if P_M is uniformly continuous.*

Remark. In the particular case that M is a Chebyshev subspace, this corollary was established by Holmes and Kripke [8] by a similar argument.

If the metric projection onto a *finite*-dimensional subspace is (pointwise) Lipschitz continuous, then it admits a selection with the same continuity property. This is a consequence of the following more general result.

2.3. PROPOSITION. *Let Y be a finite-dimensional subspace of X and $F: X \to \mathscr{K}(Y)$. If F is Lipschitz (resp. Pointwise Lipschitz) continuous, then F has a selection which is Lipschitz (resp. pointwise Lipschitz) continuous.*

Remark. Proposition 2.3 was proved by Przeslawski [14] and Dommisch [5] in the case where F is Lipschitz continuous and $Y = \mathbb{R}^n$. In general, by using the Steiner point [15] of a compact convex set in $\mathbb{R}^n$, we can easily prove Proposition 2.3. Here we outline only the idea of the proof since the details are readily verified.

Let $\dim Y = n$ and let φ be the isomorphism between Y and $\mathbb{R}^n$. Let $\sigma: \mathscr{K}(\mathbb{R}^n) \to \mathbb{R}^n$ denote the "Steiner map" [15]. Then $s = \varphi^{-1} \circ \sigma \circ \varphi \circ F$ is a Lipschitz (resp. pointwise Lipschitz) continuous selection of F. Moreover, we have

$$\|s(x) - s(y)\| \leqslant \|\varphi^{-1}\| \cdot n \cdot \|\varphi\| \cdot \lambda(x) \, \|x - y\|$$

for all $x, y \in X$, where $\lambda(x)$ denotes the Lipschitz constant of F at x.

The first author is indebted to Joram Lindenstrauss for introducing him to Steiner points which resulted in Proposition 2.3.

2.4. Corollary. *Let M be a finite-dimensional subspace of X. If P_M is Lipschitz* (resp. *pointwise Lipschitz*) *continuous, then P_M has a selection which is Lipschitz* (resp. *pointwise Lipschitz*) *continuous.*

Holmes and Kripke [8] made the following conjecture.

Conjecture. If X is strictly convex and reflexive and there exists a constant $\lambda > 0$ such that for each closed convex set K in X,

$$\|P_K(x) - P_K(y)\| \leqslant \lambda \, \|x - y\| \qquad (x, y \in X), \tag{*}$$

then X must be isomorphic to Hilbert space.

The next theorem and corollary show in particular that the Holmes–Kripke conjecture is true. In fact, it is true under somewhat weaker hypotheses.

2.5. Theorem. *Let X be a reflexive Banach space and suppose that the metric projection onto each closed subspace has a Lipschitz continuous metric selection. Then X is isomorphic to Hilbert space.*

Proof. By a result of Lindenstrauss [10, Corollary 1 of Theorem 3], each closed subspace must be complemented. By the complemented subspace theorem of Lindenstrauss and Tzafriri [11], the result follows. ∎

2.6. Corollary. *Let X be a reflexive and strictly convex Banach space. If each closed subspace has a Lipschitz continuous metric projection, then X is isomorphic to Hilbert space.*

This corollary clearly substantiates the Holmes–Kripke conjecture. In fact, for X to be isomorphic to Hilbert space, it is only necessary that (*) hold for each closed *subspace* K (and not every closed convex set K) and the constant λ in (*) may depend on the approximating subspace K (and not be universal for all K).

3. Characterizations of Continuous, Pointwise Lipschitz Continuous, and Lipschitz Continuous Metric Selections

Our first result shows that for a general class of set-valued mappings (which includes metric projections), the existence of a selection satisfying any one of the three continuity properties being considered is equivalent to the existence of one which is also homogeneous.

3.1. Lemma. *Let $F: X \to \mathscr{K}(Y)$ be bounded and homogeneous. Then the following statements are equivalent:*

(1) *F has a (continuous, pointwise Lipschitz continuous, Lipschitz continuous) selection;*

(2) *F has a (continuous, pointwise Lipschitz continuous, Lipschitz continuous) selection which is homogeneous.*

Proof. It suffices to prove (1) ⇒ (2). Let F have a (continuous, pointwise Lipschitz continuous, Lipschitz continuous) selection f. Define $\tilde{f}$ on X by

$$\tilde{f}(x) = \begin{cases} \frac{1}{2}\,\|x\|\,[f(x/\|x\|) - f(-x/\|x\|)] & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Clearly, $\tilde{f}$ is odd (i.e., $\tilde{f}(-x) = -\tilde{f}(x)$) and

$$\|\tilde{f}(x)\| \leqslant c\,\|x\|, \qquad x \in X,$$

since $\|f(x)\| \leqslant c\,\|x\|$. (Here c is the constant of Relation (1.3) appearing in the definition of boundedness of F.)

Since F is homogeneous, $F(0) = \{0\}$ so $\tilde{f}(0) = 0 \in F(0)$. If $x \neq 0$, then

$$\begin{aligned} \tilde{f}(x) &= \tfrac{1}{2}\,\|x\|\,f(x/\|x\|) - \tfrac{1}{2}\,\|x\|\,f(-x/\|x\|) \\ &\in \tfrac{1}{2}\,\|x\|\,F(x/\|x\|) - \tfrac{1}{2}\,\|x\|\,F(-x/\|x\|) \\ &= \tfrac{1}{2}F(x) + \tfrac{1}{2}F(x) = F(x) \end{aligned}$$

since $F(x)$ is convex. Thus $\tilde{f}$ is an odd selection for F.

Next suppose $x \neq 0$. If $\alpha > 0$, then

$$\begin{aligned} \tilde{f}(\alpha x) &= \frac{1}{2}\,\|\alpha x\|\,[f(\alpha x/\|\alpha x\|) - f(-\alpha x/\|\alpha x\|)] \\ &= \frac{\alpha}{2}\,\|x\|\,[f(x/\|x\|) - f(-x/\|x\|)] = \alpha\tilde{f}(x). \end{aligned}$$

If $\alpha < 0$, then $-\alpha > 0$ and since $\tilde{f}$ is odd, we get

$$\tilde{f}(\alpha x) = \tilde{f}((-\alpha)(-x)) = -\alpha \tilde{f}(-x) = \alpha \tilde{f}(x).$$

Thus $\tilde{f}$ is a homogeneous selection for F.

It remains to show that f is (continuous, pointwise Lipschitz continuous, Lipschitz continuous).

Assume first that f is pointwise Lipschitz continuous. Then for each $x \in X$ there exists $\lambda(x) > 0$ such that

$$\|f(x) - f(y)\| \leqslant \lambda(x) \|x - y\|, \qquad y \in X.$$

We will show that $\tilde{f}$ is pointwise Lipschitz continuous.

Fix any $x \in X \setminus \{0\}$. Then for each $y \in X \setminus \{0\}$, we have

$$\begin{aligned}
\|f(x/\|x\|) - f(y/\|y\|)\| &\leqslant \lambda(x/\|x\|) \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \\
&\leqslant \lambda(x/\|x\|) \left[\frac{1}{\|x\|} \|x - y\| + \left| \frac{1}{\|x\|} - \frac{1}{\|y\|} \right| \|y\| \right] \\
&\leqslant 2\lambda(x/\|x\|) \frac{1}{\|x\|} \|x - y\|.
\end{aligned}$$

Replacing x and y with their negatives in this inequality, we obtain

$$\|f(-x/\|x\|) - f(-y/\|y\|)\| \leqslant 2\lambda(-x/\|x\|) \frac{1}{\|x\|} \|x - y\|.$$

From these two inequalities, we deduce

$$\begin{aligned}
&\|\tilde{f}(x/\|x\|) - \tilde{f}(y/\|y\|)\| \\
&\quad \leqslant \frac{1}{2} \|f(x/\|x\|) - f(y/\|y\|)\| + \frac{1}{2} \|f(-x/\|x\|) - f(-y/\|y\|)\| \\
&\quad \leqslant [\lambda(x/\|x\|) + \lambda(-x/\|x\|)] \frac{1}{\|x\|} \|x - y\|.
\end{aligned}$$

Finally, using the latter inequality, we obtain

$$\begin{aligned}\|\tilde{f}(x)-\tilde{f}(y)\| &= \left\| \|x\| \tilde{f}\left(\frac{x}{\|x\|}\right) - \|y\| \tilde{f}\left(\frac{y}{\|y\|}\right)\right\| \\ &\leqslant \|x\| \left\| \tilde{f}\left(\frac{x}{\|x\|}\right) - \tilde{f}\left(\frac{y}{\|y\|}\right)\right\| + | \, \|x\| - \|y\| \, | \left\| \tilde{f}\left(\frac{y}{\|y\|}\right)\right\| \\ &\leqslant \left[\lambda\left(\frac{x}{\|x\|}\right) + \lambda\left(\frac{-x}{\|x\|}\right)\right] \|x-y\| + c\,\|x-y\| \\ &= \tilde{\lambda}(x)\,\|x-y\|,\end{aligned}$$

where $\tilde{\lambda}(x) := \lambda(x/\|x\|) + \lambda(-x/\|x\|) + c$. Also,

$$\|\tilde{f}(x) - \tilde{f}(0)\| = \|\tilde{f}(x)\| \leqslant c\,\|x\| \leqslant \tilde{\lambda}(x)\,\|x\|.$$

Thus $\tilde{f}$ is pointwise Lipschitz continuous at x with constant $\tilde{\lambda}(x)$.

Since for all $y \in X$,

$$\|\tilde{f}(0) - \tilde{f}(y)\| = \|\tilde{f}(y)\| \leqslant c\,\|y\| =: \tilde{\lambda}(0)\,\|y\|,$$

we see that $\tilde{f}$ is pointwise Lipschitz continuous.

This argument also proves that $\tilde{f}$ is Lipschitz continuous (with Lipschitz constant $\tilde{\lambda} = 2\lambda + c$) whenever f is Lipschitz continuous (with constant λ).

Finally, the argument that $\tilde{f}$ is continuous if f is continuous is a simple exercise. ▮

Remark. It is worth noting that if $S = S(X) = \{x \in X \mid \|x\| = 1\}$, our proof actually shows that Statement (2) is equivalent to

(1′) $F|_S$ has a (continuous, pointwise Lipschitz continuous, Lipschitz continuous) selection.

Let M be a proximinal subspace of X. The quotient space X/M is normed as usual by

$$\|x + M\| := d(x, M).$$

Let $F: X \to \mathscr{K}(X)$ be a "submap" of P_M (i.e., $F(x) \subset P_M(x)$ for every x) which is additive modulo M. We define a mapping $\tilde{F}$ on X/M by

$$\tilde{F}(x+M) := x - F(x), \qquad x \in X.$$

(To see that $\tilde{F}$ is well-defined, let $x + M = y + M$. Then $m := x - y \in M$ and since F is additive modulo M,

$$x - F(x) = y + m - F(y+m) = y - F(y).)$$

The next technical lemma is the key element in the main results of this section.

3.2. LEMMA. *Let M be a proximinal subspace of X and suppose that $F\colon X \to \mathscr{H}(X)$ is a submap of P_M which is homogeneous and additive modulo M. Then*

(1) *$\tilde{F}$ is homogeneous, bounded, and $\tilde{F}\colon X/M \to \mathscr{H}(X)$.*

(2) *F has a (continuous, pointwise Lipschitz continuous, Lipschitz continuous) selection which is homogeneous and additive modulo M if and only if $\tilde{F}$ has a (continuous, pointwise Lipschitz continuous, Lipschitz continuous) selection.*

(3) *F is lower semicontinuous if and only if $\tilde{F}$ is lower semicontinuous.*

Proof. (1) Since F is homogeneous, for any $\alpha \in \mathbb{R}$,

$$\begin{aligned}\tilde{F}[\alpha(x+M)] &= \tilde{F}(\alpha x + M) = \alpha x - F(\alpha x)\\ &= \alpha[x - F(x)] = \alpha \tilde{F}(x+M)\end{aligned}$$

so $\tilde{F}$ is homogeneous. To see that $\tilde{F}$ is bounded, let $x \in X$ and $y \in \tilde{F}(x+M)$. Then $y = x - y_0$ for some $y_0 \in F(x) \subset P_M(x)$. Hence

$$\|y\| = \|x - y_0\| = \|x + M\|.$$

That is,

$$\sup\{\|y\| \mid y \in \tilde{F}(x+M)\} = \|x+M\|$$

and $\tilde{F}$ is bounded. Finally, since $F(x) \in \mathscr{H}(X)$, $\tilde{F}(x+M) \in \mathscr{H}(X)$.

(2) Suppose F has a (continuous, pointwise Lipschitz continuous, Lipschitz continuous) selection f which is homogeneous and additive modulo M. Define $\tilde{f}$ on X/M by

$$\tilde{f}(x+M) = x - f(x), \qquad x \in X.$$

By (1), $\tilde{f}$ is well-defined, bounded, homogeneous, and $\tilde{f}\colon X/M \to \mathscr{H}(X)$. Furthermore, $\tilde{f}$ is a selection for $\tilde{F}$.

If f is pointwise Lipschitz continuous, then for each $x \in X$ there exists $\lambda(x) > 0$ such that

$$\|f(x) - f(y)\| \leqslant \lambda(x)\, \|x - y\|, \qquad y \in X.$$

Then, for any $m \in M$,

$$\begin{aligned}\|\tilde{f}(x+M)-\tilde{f}(y+M)\| &= \|x-f(x)-(y-f(y))\| \\ &= \|x-y-m-[f(x)-f(y+m)]\| \\ &\leqslant \|x-y-m\| + \|f(x)-f(y+m)\| \\ &\leqslant [1+\lambda(x)]\ \|x-y-m\|.\end{aligned}$$

Taking the infimum over all $m \in M$, we obtain

$$\begin{aligned}\|\tilde{f}(x+M)-\tilde{f}(y+M)\| &\leqslant (1+\lambda(x))\, d(x-y, M) \\ &= (1+\lambda(x))\ \|x+M-(y+M)\|.\end{aligned}$$

Thus $\tilde{f}$ is pointwise Lipschitz continuous. In particular, if f is Lipschitz continuous, so is $\tilde{f}$.

Now suppose f is continuous and $x_n + M \to x + M$, i.e., $d(x_n - x, M) \to 0$. Select $m_n \in M$ so that $x_n - x - m_n \to 0$ or $x_n - m_n \to x$. Then

$$\begin{aligned}\tilde{f}(x_n+M) &= x_n - f(x_n) = x_n - m_n - f(x_n - m_n) \\ &\to x - f(x) = \tilde{f}(x+M)\end{aligned}$$

implies that $\tilde{f}$ is continuous.

For the converse, let $\tilde{f}$ be a (continuous, pointwise Lipschitz continuous, Lipschitz continuous) selection for $\tilde{F}$. By Lemma 3.1, we may assume $\tilde{f}$ is homogeneous. Define f on X by

$$f(x) := x - \tilde{f}(x+M), \qquad x \in X.$$

Then f is a selection for F which is homogeneous. Further, for any $m \in M$,

$$\begin{aligned}f(x+m) &= x+m-\tilde{f}(x+m+M) = x+m-\tilde{f}(x+M) \\ &= f(x)+m\end{aligned}$$

so f is additive modulo M.

The proof that f is (continuous, pointwise Lipschitz continuous, Lipschitz continuous) is similar to the first part of the proof.

(3) Let F be lower semicontinuous. Then for any $x \in X$, $y \in F(x)$, and $x_n \to x$, we have that $d(y, F(x_n)) \to 0$. To show that $\tilde{F}$ is lower semicontinuous, let $x \in X$, $x_n + M \to x + M$, and $y \in \tilde{F}(x+M)$. We need to verify that $d(y, \tilde{F}(x_n + M)) \to 0$. Select $m_n \in M$ such that $x_n - m_n \to x$. Then $y = x - y_0$ for some $y_0 \in F(x)$, so

$$\begin{aligned}d(y, \tilde{F}(x_n+M)) &= d(y, x_n - F(x_n)) \\ &= d(x-y_0, x_n - m_n - F(x_n - m_n)) \\ &\leqslant \|x_n - m_n - x\| + d(y_0, F(x_n - m_n)) \\ &\to 0 \text{ as } n \to \infty.\end{aligned}$$

Conversely, let $\tilde{F}$ b lower semicontinuous, $x_n \to x$, and $y \in F(x)$. Then $x_n + M \to x + M$ and $x - y \in \tilde{F}(x + M)$ implies that

$$\begin{aligned} d(y, F(x_n)) &= d(-y, -F(x_n)) = d(x - y, x - F(x_n)) \\ &= d(x - y, x - x_n + x_n - F(x_n)) \\ &= d(x - y, x - x_n + \tilde{F}(x_n + M)) \\ &\leqslant \|x - x_n\| + d(x - y, \tilde{F}(x_n + M)) \to 0, \end{aligned}$$

so F is lower semicontinuous. ▮

A subset N of X is called *homogeneous* if $\alpha N \subset N$ for each $\alpha \in \mathbb{R}$. If M is a proximinal subspace of X and N is a subset (not necessarily a subspace) of X, we will write

$$X = M \oplus N$$

to mean that each $x \in X$ has a unique representation as $x = m + n$, where $m \in M$ and $n \in N$.

Recall that the quotient map $Q = Q_M : X \to X/M$, defined by $Q(x) = x + M$, is linear, $\|Qx\| \leqslant \|x\|$ for every x, and $\|Qx\| = \|x\|$ for each $x \in \ker P_M$. In particular, for any subset N of X, the restriction mapping $Q|_N$ is Lipschitz continuous.

A homeomorphism f between two metric spaces is called a *Lipschitz* (resp. *pointwise Lipschitz*) *homeomorphism* provided that both f and f^{-1} are Lipschitz (resp. pointwise Lipschitz) continuous.

We can now characterize when the metric projection has a selection having one of three continuity properties and which is also homogeneous and additive modulo M.

3.3. THEOREM. *For a proximinal subspace M of the normed linear space X, the following statements are equivalent:*

(1) *P_M has a (continuous, pointwise Lipschitz continuous, Lipschitz continuous) selection which is homogeneous and additive modulo M;*

(2) *$\ker P_M$ contains a closed homogeneous subset N such that $X = M \oplus N$ and the mapping $p(m + n) = m$ is (continuous, pointwise Lipschitz continuous, Lipschitz continuous);*

(3) *$\ker P_M$ contains a closed homogeneous subset N such that $Q|_N$ is a (homeomorphism, pointwise Lipschitz homeomorphism, Lipschitz homeomorphism) between N and X/M.*

Moreover, the desired selection is given by p if (2) *holds and by $x \mapsto x - (Q|_N)^{-1}(x + M)$ if* (3) *holds.*

Proof. (1) $\Rightarrow$ (2). Let p be a (continuous, pointwise Lipschitz continuous, Lipschitz continuous) selection for P_M which is homogeneous and additive modulo M. Let $N = p^{-1}(0)$. Then N is closed, homogeneous, and $N \subset \ker P_M$. Also, for each $x \in X$,

$$p(x - p(x)) = p(x) - p(x) = 0,$$

so $x - p(x) \in N$ and $x = p(x) + (x - p(x))$. This shows that $X = M + N$. If $x = m + n$ for some $m \in M$ and $n \in N$, then

$$p(x) = p(m+n) = p(n) + m = m$$

and $n = x - p(x)$. Thus the representation of x is unique and hence $X = M \oplus N$. Since the mapping $m + n \mapsto m$ is just p, Statement (2) follows.

(2) $\Rightarrow$ (3). Suppose $\ker P_M$ contains a closed homogeneous subset N such that $X = M \oplus N$ and the map $p(m+n) = n$ is (continuous, pointwise Lipschitz continuous, Lipschitz continuous). First note that p is homogeneous and additive modulo M. Also, $x - p(x) \in N$ for every x so

$$\|x - p(x)\| = d(x - p(x), M) = d(x, M).$$

That is, p is a selection for P_M. By Part (2) of Lemma 3.2, with $F = p$, we see that $\tilde{p}$ is (continuous, pointwise Lipschitz continuous, Lipschitz continuous).

Claim. $Q|_N : N \to X/M$ is bijective and $(Q|_N)^{-1} = \tilde{p}$.

Assuming the claim is true, then since $Q|_N$ is Lipschitz continuous, Statement (3) follows. Thus it remains to verify the claim.

To verify that $Q|_N$ is injective, let $n_i \in N$ $(i = 1, 2)$ and $Q(n_1) = Q(n_2)$. Then $n_1 + M = n_2 + M$ so $m := n_1 - n_2 \in M$ and $n_1 = m + n_2$. By uniqueness of the representation for n_1, $n_1 = n_2$ and $m = 0$. Thus $Q|_N$ is injective. For any $x \in X$, $x = m + n$ for some $m \in M$, $n \in N$. Thus

$$x + M = n + M = Q(n),$$

so $Q|_N$ is surjective, hence bijective.

Next note that for any $x \in X$, $x - p(x) \in N$ and

$$Q(x - p(x)) = x - p(x) + M = x + M.$$

Hence

$$(Q|_N)^{-1}(x + M) = x - p(x) = \tilde{p}(x + M).$$

That is, $(Q|_N)^{-1} = \tilde{p}$ and the claim is verified.

(3)⇒(1). Suppose that ker P_M contains a closed homogeneous subset N such that $Q|_N$ is a (homeomorphism, pointwise Lipschitz homeomorphism, Lipschitz homeomorphism) between N and X/M. Define

$$p(x) := x - (Q|_N)^{-1}(x+M), \qquad x \in X.$$

Claim. p is a selection for P_M which is homogeneous and additive modulo M.

Assuming the claim is true for the moment, we see that $\tilde{p} = (Q|_N)^{-1}$. Hence by Part (2) of Lemma 3.2, we deduce that p is (continuous, pointwise Lipschitz continuous, Lipschitz continuous). Therefore, to establish (1), it suffices to verifiy the claim.

For any $x \in X$, $x - p(x) = (Q|_N)^{-1}(x+M) \in N$ and, for any $m \in M$,

$$\begin{aligned} p(x+m) &= x + m - (Q|_N)^{-1}(x+m+M) \\ &= x + m - (Q|_N)^{-1}(x+M) = p(x) + m, \end{aligned}$$

so p is additive modulo M. Since $x - p(x) \in N$,

$$Q(x - p(x)) = x - p(x) + M$$

implies

$$(Q|_N)^{-1}(x - p(x) + M) = x - p(x) = (Q|_N)^{-1}(x+M).$$

Since $Q|_N$ is injective, $x - p(x) + M = x + M$, so $p(x) \in M$. Also,

$$\|x - p(x)\| = d(x - p(x), M) = d(x, M)$$

implies $p(x) \in P_M(x)$; i.e., p is a selection for P_M. Finally, the homogeneity of N implies that $(Q|_N)^{-1}$, hence p, is homogeneous. This proves the claim.

The last statement of the theorem was established during the course of the proof. ∎

We can now state and prove the two main theorems of this section.

3.4. THEOREM. *Let M be a proximinal subspace of a Banach space X. Then the following statements are equivalent*:

(1) *P_M has a continuous selection*;

(2) *P_M has a continuous selection which is homogeneous and additive modulo M*;

(3) ker P_M *contains a closed homogeneous subset N such that $X = M \oplus N$ and the mapping $p(m+n) = m$ is continuous*;

(4) ker P_M *contains a closed homogeneous subset* N *such that* $Q|_N$ *is a homeomorphism between* N *and* X/M.

Moreover, the continuous selection is given by p *if* (3) *holds and by* $x \mapsto x - (Q|_N)^{-1}(x+M)$ *if* (4) *holds.*

Proof. By Theorem 3.3, it suffices to prove the implication (1) ⇒ (2). Thus, suppose P_M has a continuous selection. Define F on X by

$$F(x) := \{p(x) \mid p \text{ is a continuous selection for } P_M\}.$$

It was noted in [4] that F is the maximal lower semicontinuous submap of P_M. In particular, $F(x) \in \mathscr{H}(X)$ for every x.

Claim. F is homogeneous and additive modulo M.

Assuming the claim for the moment, it follows by Part (3) of Lemma 3.2 that $\tilde{F}: X/M \to \mathscr{H}(X)$ is lower semicontinuous. By the Michael selection theorem [12], $\tilde{F}$ has a continuous selection. By Part (2) of Lemma 3.2, F has a continuous selection which is homogeneous and additive modulo M. Since F is a submap of P_M, this selection is also a selection for P_M. That is, (2) holds. Thus it remains to prove the claim.

To prove F is homogeneous, fix any $\alpha \in \mathbb{R}$, $\alpha \neq 0$. Note that for any function $p: X \to X$, we can define $p': X \to X$ by

$$p'(x) := \frac{1}{\alpha} p(\alpha x), \qquad x \in X.$$

It is easy to verify that p is a continuous selection for P_M if and only if p' is. Thus

$$\begin{aligned} F(\alpha x) &= \{p(\alpha x) \mid p \text{ is a continuous selection for } P_M\} \\ &= \{\alpha p'(x) \mid p' \text{ is a continuous selection for } P_M\} \\ &= \alpha\{p'(x) \mid p' \text{ is a continuous selection for } P_M\} \\ &= \alpha F(x) \end{aligned}$$

implies F is homogeneous.

To show F is additive modulo M, fix any $m \in M$. Again note that a function $p: X \to X$ is a continuous selection for P_M if and only if the function $p'': X \to X$, defined by

$$p''(x) := p(x+m) - m, \qquad x \in X,$$

is a continuous selection for P_M. Thus

$$\begin{aligned} F(x+m) &= \{p(x+m) \mid p \text{ is a continuous selection for } P_M\} \\ &= \{p''(x)+m \mid p'' \text{ is a continuous selection fr } P_M\} \\ &= \{p''(x) \mid p'' \text{ is a continuous selection for } P_M\} + m \\ &= F(x)+m, \end{aligned}$$

so F is additive modulo M. This proves the claim. ∎

Fakhouri [6] has proved a related result: P_M has a continuous selection if and only if the map $x \mapsto (x+M) \cap \ker P_M$ has a continuous selection which is homogeneous.

3.5. THEOREM. *Let M be a proximinal subspace which is complemented in the normed linear space X. Then the following statements are equivalent*:

(1) *P_M has a (pointwise) Lipschitz continuous selection*;

(2) *P_M has a (pointwise) Lipschitz continuous selection which is homogeneous and additive modulo M*;

(3) $\ker$ *P_M contains a closed homogeneous subset N such that $X = M \oplus N$ and the mapping $p(m+n) = m$ is (pointwise) Lipschitz continuous*;

(4) $\ker$ *P_M contains a closed homogeneous subset N such that $Q|_N$ is a (pointwise) Lipschitz homeomorphism between N and X/M.*

Moreover, the desired selection is given by p if (3) *holds and by $x \mapsto x - (Q|_N)^{-1}(x+M)$ if* (4) *holds.*

Proof. By Theorem 3.3, it suffices to verify the implication (1) ⇒ (2). Since M is complemented, there exist a closed subspace L in X and a linear projection P onto M along L. Thus $X = M \oplus L$. By Lemma 3.1, the mapping $F = P_M|_L$ has a (pointwise) Lipschitz continuous selection f which is homogeneous. Define p on X by $p = f \circ (I-P) + P$. It is a simple exercise to verify that p is a (pointwise) Lipschitz continuous selection for P_M which is homogeneous and additive modulo M. ∎

Theorem 3.4 can be strengthened in the particular cases when M is finite-dimensional or finite-codimensional.

3.6. THEOREM. *Let M be a finite-dimensional subspace of the Banach space X. Then P_M has a continuous selection if and only if* $\ker$ *P_M contains a closed homogeneous subset N with $X = M \oplus N$.*

Proof. By Theorem 3.4, it suffices to verify that if $\ker P_M$ contains a closed homogeneous subset N with $X = M \oplus N$, then the mapping $p: X \to M$ defined by $p(x) = m$, where $x = m + n$, is continuous.

We first note that p is a selection for P_M. In particular, $\|p(x)\| \leqslant 2 \|x\|$ for all x. Fix any $x \in X$ and let $x_k \to x$. Since $\{p(x_k)\}$ is a bounded sequence in the finite-dimensional space M, every subsequence of $\{p(x_k)\}$ has a subsequence which converges:

$$p(x_{k_j}) \to m \in M. \tag{3.6.1}$$

Then $x_{k_j} - p(x_{k_j}) \to x - m =: n \in N$ since N is closed. Since $x = m + n$ and this representation is unique, $m = p(x)$ and $n = x - p(x)$. By (3.6.1), $p(x_{k_j}) \to p(x)$. It follows that $p(x_k) \to p(x)$ and p is continuous at x. ∎

Recall that a set N is called *boundedly compact* if each bounded sequence in N has a subsequence which converges to a point in N.

3.7. THEOREM. *Let M be a proximinal subspace having finite-codimension in the Banach space X. Then P_M has a continuous selection if and only if* ker P_M *contains a boundedly compact homogeneous subset N with $X = M \oplus N$.*

Proof. Suppose P_M has a continuous selection. By Theorem 3.4, ker P_M contains a closed homogeneous subset N such that $Q|_N$ is a (norm-preserving) homeomorphism between N and X/M. Using the one-to-oneness of $Q|_N$, it is easy to verify that $X = M \oplus N$. Since $\dim X/M = \operatorname{codim} M < \infty$, each bounded sequence in X/M has a convergent subsequence. It follows that each bounded sequence in N must have a subsequence converging to a point in N. That is, N is boundedly compact.

Conversely, suppose $\ker P_M$ contains a boundedly compact homogeneous subset N with $X = M \oplus N$. By the same proof as in Theorem 3.6, the mapping $q: X \to N$ defined by $q(x) = n$, $x = m + n$, is continuous. Thus $p := I - q$ is also continuous and $p(m+n) = m$. By Theorem 3.4, P_M has a continuous selection. ∎

These results can be further sharpened when M is a Chebyshev subspace. For this, it is convenient to first make the following observation.

3.8. LEMMA. *If M is a Chebyshev subspace, $N \subset \ker P_M$, and $X = M \oplus N$, then $N = \ker P_M$.*

Proof. If not, choose $y \in \ker P_M \backslash N$. Then $y = m + n$ for some $m \in M$ and $n \in N$. Hence

$$0 = P_M(y) = m + P_M(n) = m$$

so $y = n \in N$, a contradiction. ∎

Next recall the well-known result of Cheney and Wulbert.

THEOREM [1]. *A closed subspace M is Chebyshev if and only if* $X = M \oplus \ker P_M$.

3.9. COROLLARY. *Let M be a Chebyshev subspace which is complemented in the Banach space X. Then the following statements are equivalent*:

(1) P_M *is (continuous, pointwise Lipschitz continuous, Lipschitz continuous)*;

(2) $Q|_{\ker P_M}$ *is a (homeomorphism, pointwise Lipschitz homeomorphism, Lipschitz homeomorphism) between* $\ker P_M$ *and* X/M.

Proof. This follows by Combining Theorems 3.4 and 3.5, Lemma 3.8, and the Cheney Wulbert theorem. ∎

That part of Corollary 3.9 pertaining to continuous selections (viz. P_M is continuous if and only if $Q|_{\ker P_M}$ is a homeomorphism) was first established by Holmes [7].

3.10. COROLLARY (Cheney and Wulbert [1]). *Let M be a Chebyshev subspace of finite codimension in the Banach space X. Then P_M is continuous if and only if $\ker P_M$ is boundedly compact.*

Proof. This follows from Theorem 3.7, Lemma 3.8, and the Cheney-Wulbert theorem. ∎

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